MATH 103B – Discussion Worksheet 8
June 8, 2023

Topic: Galois theory (Judson Chapter 23.1-2)
Announcements:

1. The final exam will take place on Wednesday June 14 3-4:30 pm.
2. Steve will hold office hours next Tuesday and Wednesday 10:30-12:30 at AP&M 5412. Please email (yoh011@ucsd.edu) in advance if you are coming.

Example 0.1. $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is *not* Galois, whereas $\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q}$ is Galois with $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q} \cong S_3$.

Problem 1. Consider $\mathbb{Q}(\zeta_5)/\mathbb{Q}$ (recall $\zeta_5 = e^{2\pi i/5}$ is a primitive fifth root of unity). This is a Galois extension with $\mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$ generated by $\varphi(\zeta_5) = \zeta_5^2$.

a) Compute $\varphi^2(\zeta_5)$.

b) Show that φ^2 fixes $\zeta_5 + \zeta_5^4$.

c) Show $\mathbb{Q}(\zeta_5 + \zeta_5^4) = \mathbb{Q}(\zeta_5)^{\langle \varphi^2 \rangle}$.

Problem 2. Consider $\mathbb{Q}(\sqrt[3]{5}, \zeta_3)/\mathbb{Q}$. This is a Galois extension with $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{5}, \zeta_3)/\mathbb{Q}) \cong S_3$, the symmetric group on three letters. Recall $S_3 = \langle \sigma, \tau | \sigma^3 = \tau^2 = 1, \sigma\tau = \tau\sigma^{-1} \rangle$. We have

$$\sigma(\sqrt[3]{5}) = \zeta_3 \sqrt[3]{5}, \sigma(\zeta_3) = \zeta_3; \tau(\sqrt[3]{5}) = \sqrt[3]{5}, \tau(\zeta_3) = \zeta_3^2.$$

a) List all elements of S_3.

b) Find the subgroup of S_3 corresponding to the subfield $\mathbb{Q}(\zeta_3) \subseteq \mathbb{Q}(\sqrt[3]{5}, \zeta_3)$. Your answer should list the elements in the subgroup explicitly.

c) Prove directly (i.e. without using field theory) that the subgroup you obtained in Part b is normal in S_3.